

Fall 2006: MA Algebra Preliminary Exam

Instructions:

- (1) *Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*
- (2) *Use separate sheets for the solution of each problem.*

Problem 1. Let G be a matrix group, and let $g \in G$ be an element with $\det(g) \neq 1$. Show that $g \notin G'$, the commutator group of G .

Problem 2. Let $A : V \rightarrow V$ be an operator on a finite-dimensional vector space V . Suppose A has characteristic polynomial $x^2(x-1)^4$ and minimal polynomial $x(x-1)^2$. What is the dimension of V ? What are the possible Jordan forms of A ?

Problem 3. Show that $\mathbb{Z}$ is a principal ideal domain.

Problem 4. Let G denote a finite abelian group. Let us consider the set G^* of all homomorphisms of the group G into the multiplicative group $\mathbb{C}^\times$ of nonzero complex numbers.

- (a) Check that G^* can be considered as a group with respect to the operation of multiplication of homomorphisms.
- (b) Prove that the group G^* is isomorphic to the group G .

Problem 5. Let us assign to every nonsingular complex 2×2 matrix A a transformation ϕ_A of the vector space Mat_2 of complex 2×2 matrices defined by the formula

$$\phi_A(X) = AXA^{-1}.$$

- (a) Check that this formula specifies an action of the group $GL_2(\mathbb{C})$ of nonsingular complex matrices on Mat_2 ; moreover, it specifies a linear representation of this group.
- (b) Prove that this representation is reducible.

Problem 6. Consider the dihedral group D_9 (the group of isometries of regular 9-gons).

- (a) Write down a list of all elements of D_9 .
- (b) Prove that D_9 cannot be represented as a direct product of two non-trivial groups.
- (c) Determine if D_9 is solvable.

Fall 2006: MA Analysis Preliminary Exam

Instructions:

- (1) *Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.*
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Problem 1. Let $C([0, 1])$ be the Banach space of continuous real-valued functions on $[0, 1]$, with the norm $\|f\|_\infty = \sup_x |f(x)|$. Let $k : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ be a given continuous function. Let $T_k : C([0, 1]) \rightarrow C([0, 1])$ be the linear operator given by $T_k(f)(x) = \int_0^1 k(x, y)f(y) dy$.

- (a) Show that T_k is a bounded operator.
- (b) Find an expression for $\|T_k\|$ in terms of k .
- (c) What is $\|T_k\|$ if $k(x, y) = x^2y^3$?

Problem 2. Let X be a metric space.

- (a) Define X is sequentially compact.
- (b) Define X is a complete metric space.
- (c) Prove that a sequentially compact metric space X is complete.
- (d) Let $B = \{x : \|x\|_2 \leq 1\}$ be the unit ball in $\ell^2(\mathbb{N})$. Show that B is not sequentially compact.

Problem 3. Give an example of a Banach space X and a sequence (x_n) of elements in X such that $\sum_{n=1}^\infty x_n$ converges unconditionally (converges regardless of order), but does not converge absolutely ($\sum_{n=1}^\infty |x_n|$ does not converge). Prove this.

Problem 4. Let $f \in L^2(\mathbb{T})$, and let $(\hat{f}_n)_{n \in \mathbb{Z}}$ be the Fourier coefficient sequence of f ; here, $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$. If $(\hat{f}_n) \in \ell^1(\mathbb{Z})$, does it follow that f is continuous? (In other words, is there a continuous function that is equivalent to f in $L^2(\mathbb{T})$?) Prove your assertion.

Problem 5. Find all solutions T of the equation $x^{2006}T = 0$ in the space of tempered distributions $S^*(\mathbb{R}^1)$.

Problem 6. In which of the following cases is the operator $A = i\frac{d}{dx}$ acting on $L^2([0, 1])$ symmetric, essentially self-adjoint, self-adjoint? Justify your answers.

- (a) $D_A = C^1[0, 1]$
(the space of continuously differentiable complex-valued functions on $[0, 1]$)
- (b) $D_A = \{f \in C^1[0, 1] : f(0) = f(1)\}$
- (c) $D_A = \{f \in C^1[0, 1] : f(0) = f(1) = 0\}$