

Fall 2007: MA Algebra Preliminary Exam

Instructions:

1. All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.
2. Use separate sheets for the solution of each problem.

Problem 1. Suppose that ρ is a complex matrix representation of a finite group G . Show that every matrix $\rho(g)$ is diagonalizable.

Problem 2. Consider the ring $\mathbb{R}[[x]]$ of formal power series in x with real coefficients. Namely, this is the set of all infinite series $a_0 + a_1x + a_2x^2 + \dots$ with no conditions on convergence. What are the units (invertible elements) in this ring? What are the ideals?

Problem 3. If H is a subgroup of a group G , then the normalizer $N(H)$ of H is defined as the set of g in G such that $gHg^{-1} = H$. It is the largest subgroup of G that contains H as a normal subgroup. Let G be the symmetric group S_7 ; let $H \subset G$ be the cyclic subgroup generated by a 7-cycle.

Find the number of elements of the normalizer $N(H)$ of H in G .

Problem 4. Compute the number of groups of order ≤ 1029 each of which contains exactly three elements of order 3.

Problem 5. Show that the group $\mathbb{Q}$ of rational numbers (with respect to the addition operation) is not finitely generated.

Problem 6. Show that

$$\det(\exp(A)) = e^{\text{tr}(A)}$$

for every complex $n \times n$ matrix A , where $\exp(A)$ is defined as

$$\exp(A) = 1 + A + \frac{A^2}{2} + \dots + \frac{A^k}{k!} + \dots$$