

Fall 2008: MA Analysis Preliminary Exam

Instructions:

1. All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.
2. Use separate sheets for the solution of each problem.

Problem 1: Prove that the dual space of c_o is ℓ^1 , where

$$c_o = \{x = (x_n) \text{ such that } \lim x_n = 0\}.$$

Problem 2: Let $\{f_n\}$ be a sequence of differentiable functions on a finite interval $[a, b]$ such that the functions themselves and their derivatives are uniformly bounded on $[a, b]$. Prove that $\{f_n\}$ has a uniformly converging subsequence.

Problem 3: Let $f \in L^1(\mathbb{R})$ and V_f be the closed subspace generated by the translates of f : $\{f(\cdot - y) \mid \forall y \in \mathbb{R}\}$. Suppose $\hat{f}(\xi_0) = 0$ for some ξ_0 . Show that $\hat{h}(\xi_0) = 0$ for all $h \in V_f$. Show that if $V_f = L^1(\mathbb{R})$, then $\hat{f}$ never vanishes.

Problem 4: (a) State the Stone-Weierstrass theorem for a compact Hausdorff space X .

(b) Prove that the algebra generated by functions of the form $f(x, y) = g(x)h(y)$ where $g, h \in C(X)$ is dense in $C(X \times X)$.

Problem 5: For $r > 0$, define the dilation $d_r f : \mathbb{R} \rightarrow \mathbb{R}$ of a function $f : \mathbb{R} \rightarrow \mathbb{R}$ by $d_r f(x) = f(rx)$, and the dilation $d_r T$ of a distribution $T \in \mathcal{D}'(\mathbb{R})$ by

$$\langle d_r T, \phi \rangle = \frac{1}{r} \langle T, d_{1/r} \phi \rangle \quad \text{for all test functions } \phi \in \mathcal{D}(\mathbb{R}).$$

(a) Show that the dilation of a regular distribution T_f , given by

$$\langle T_f, \phi \rangle = \int f(x) \phi(x) dx,$$

agrees with the dilation of the corresponding function f .

(b) A distribution is homogeneous of degree n if $d_r T = r^n T$. Show that the δ -distribution is homogeneous of degree -1 .

(c) If T is a homogeneous distribution of degree n , prove that the derivative T' is a homogeneous distribution of degree $n - 1$.

Problem 6: Let $\ell^2(\mathbb{N})$ be the space of square-summable, real sequences $x = (x_1, x_2, x_3, \dots)$ with norm

$$\|x\| = \left(\sum_{n=1}^{\infty} x_n^2 \right)^{1/2}.$$

Define $F : \ell^2(\mathbb{N}) \rightarrow \mathbb{R}$ by

$$F(x) = \sum_{n=1}^{\infty} \left\{ \frac{1}{n} x_n^2 - x_n^4 \right\}$$

(a) Prove that F is differentiable at $x = 0$, with derivative $F'(0) : \ell^2(\mathbb{N}) \rightarrow \mathbb{R}$ equal to zero.

(b) Show that the second derivative of F at $x = 0$,

$$F''(0) : \ell^2(\mathbb{N}) \times \ell^2(\mathbb{N}) \rightarrow \mathbb{R},$$

is positive-definite, meaning that

$$F''(0)(h, h) > 0$$

for every nonzero $h \in \ell^2(\mathbb{N})$.

(c) Show that F does not attain a local minimum at $x = 0$.