

Fall 2012: MA Algebra Preliminary Exam

Instructions:

1. All problems are worth 10 points. Explain your answers clearly. Unclear answers will not receive credit. State results and theorems you are using.
2. Use separate sheets for the solution of each problem.

Problem 1.

a) Can a vector space V over an infinite field be a finite union

$$V = \bigcup_{i=1}^k V_i,$$

where for each i , $V_i \neq V$?

b) Can the group $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$ be a union of finitely many proper subgroups?

Problem 2. Let G be an abelian group with n generators. Show that every subgroup $H \subset G$ has a generating set consisting of at most n elements.

Problem 3. Let F be a field and let $P \subset F$ be the intersection of all subfields in F . Show that:

- a) If F has characteristic 0 then $P \cong \mathbb{Q}$,
- b) If F has characteristic $p > 1$ then $P \cong \mathbb{F}_p$.

Problem 4. Let R be a commutative ring and I an ideal in R . Prove or disprove: The set $\sqrt{I} = \{a \in R : \exists n \in \mathbb{N}, n > 0, a^n \in I\}$ is an ideal.

Problem 5. Find the number of field homomorphisms $\phi: \mathbb{Q}(\sqrt[3]{2}) \rightarrow \mathbb{C}$.

Problem 6.

Consider the dihedral group $D_4 = \langle r, s : s^2 = r^4 = 1, rs = sr^{-1} \rangle$ of order 8.

- a) Find the conjugacy classes of D_4 .
- b) Find the character table of D_4 .